Coding multitype branching forests : application to the law of the total progeny.

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Introduction

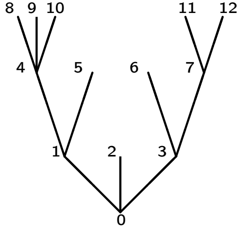
- μ distribution on $\mathbb{Z}_+$ such that $\sum_{k=0}^{\infty} k\mu(k) \leq 1$, $\mu(1) < 1$.
- τ branching (rooted) tree with offspring distribution μ.
- $O(\tau)$ total progeny of τ.
- $u_0, \ldots, u_{O(\tau)-1}$ vertices of τ ranked in the breadth first search order.
- $k_u(\tau)$ number of children of $u \in \tau$.

Introduction

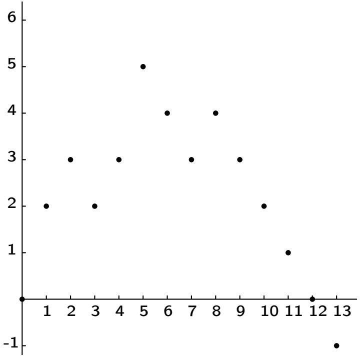
The genealogy of any tree τ is encoded through :

$$X_0 = 0\,, \quad X_{n+1}(\tau) - X_n(\tau) = k_{u_n}(\tau) - 1, \quad 0 \leq n \leq O(\tau) - 1\,.$$

$(X_n)_{n\geq 0}$ downward skip free random walk with step distribution : $\mathbb{P}(X_1 = i) = \mu(i+1)$.



Rooted tree τ



Coding walk $X(\tau)$

Introduction

The law of the total progeny $O(\tau)$ follows from the identity :

$$O(\tau) = \inf\{n : X_n = -1\}$$

and the Ballot theorem :

$$P(T_1 = n) = \frac{1}{n} P(X_n = -1) \,,$$

$T_1 = \inf\{n : X_n = -1\}$.

Theorem (Dwass, 1969)

The law of the total progeny of τ is

$$\mathbb{P}_1(O(\tau) = n) = \frac{1}{n} \mu^{*n}(n-1) \,.$$

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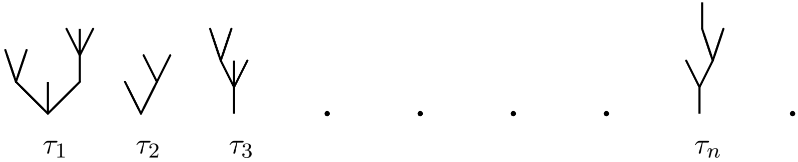


More generally, for any downward skipfree random walk (X_n),

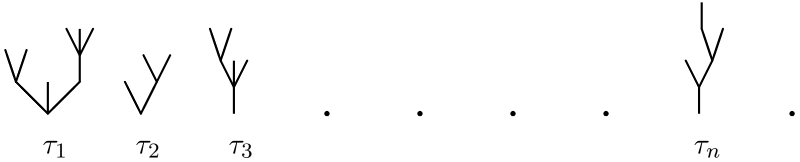
$$P(T_k = n) = \frac{k}{n} P(X_n = -k)\,,$$

where $T_k = \inf\{n : X_n = -k\}$.

- The law of the total progeny of the forest $\mathcal{F} = \{\tau_1, \ldots, \tau_k\}$ is,

$$\mathbb{P}_k(O(\mathcal{F}) = n) = \frac{k}{n} \mu^{*n}(n-k)\,.$$

Introduction



More generally, for any downward skipfree random walk (X_n),

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Progeny of 2-type branching processes

μ_1 and μ_2 probabilities on $\mathbb{Z}_+ \times \mathbb{Z}_+$.

$\mathbf{Z}_n := (Z_n^{(1)}, Z_n^{(2)})$, $n \geq 0$, 2-type branching process with progeny law (μ_1, μ_2), such that $\mathbf{Z}_0 = (1, 0)$. Assume that

$$T := \inf\{n : \mathbf{Z}_n = 0\} < \infty\,, \quad \text{a.s.}$$

What is the joint law of

$$O_1 = \sum_{n=0}^{T} Z_n^{(1)} = \text{total number of individuals of type 1 at time } T$$

$$O_2 = \sum_{n=0}^{T} Z_n^{(2)} = \text{total number of individuals of type 2 at time } T\,?$$

Progeny of 2-type branching processes

Define the mean matrix :

$$m_{ij} = \sum_{\mathbf{z} \in \mathbb{Z}_+^2} z_j \mu_i(\mathbf{z}) \,, \quad i,j \in \{1,2\} \,.$$

- $m_{12} > 0, 1 \geq m_{11} > 0$ and $m_{22} = m_{21} = 0$, (Bertoin, 2010) :

  $$\mathbb{P}_{(1,0)}(O_1 = n_1, O_2 = n_2) = \frac{1}{n_1} \mu_1^{*n_1}(n_1 - 1, n_2), \quad n_1 \geq 1,\, n_2 \geq 0 \,.$$

- $m_{12} > 0, 1 \geq m_{11}, m_{22} > 0$ but $m_{21} = 0$,

  $$\mathbb{P}_{(1,0)}(O_1 = n_1, O_2 = n_2) = \frac{1}{n_1 n_2} \sum_{j=0}^{n_2} j \mu_1^{*n_1}(n_1 - 1, j) \mu_2^{*n_2}(0, n_2 - j).$$

Progeny of 2-type branching processes

In all the remaining cases the matrix $(m_{ij})_{i,j\in\{1,2\}}$ (or the process $\mathbf{Z}$) is irreducible, i.e.

$$m_{12} > 0 \quad \text{and} \quad m_{21} > 0\,.$$

Let ρ be the dominant eigenvalue (Perron-Frobenius).

Then,

$$\rho \leq 1 \Longleftrightarrow T := \inf\{n : \mathbf{Z}_n = 0\} < \infty\,, \quad \text{a.s.}\,.$$

The process is said to be critical ($\rho = 1$) or subcritical ($\rho < 1$).

Progeny of 2-type branching processes

O_1 : total number of individuals of type 1.

O_2 : total number of individuals of type 2.

N_1 : total number of individuals of type 1 whose parent is of type 2.

N_2 : total number of individuals of type 2 whose parent is of type 1.

Theorem

Assume that $\mathbf{Z}$ is irreducible and critical or subcritical and $\mathbf{Z}_0 = (1,0)$. Then for all $n_1 \geq 1$ $n_2 \geq 0$, $1 \leq k_1 \leq n_1$ and $0 \leq k_2 \leq n_2$,

$$\mathbb{P}_{(1,0)}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2) .$$

Progeny of 2-type branching processes

O_1 : total number of individuals of type 1.

O_2 : total number of individuals of type 2.

N_1 : total number of individuals of type 1 whose parent is of type 2.

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Theorem

Assume that $\mathbf{Z}$ is irreducible and critical or subcritical and $\mathbf{Z}_0 = (1,0)$. Then for all $n_1 \geq 1$ $n_2 \geq 0$, $1 \leq k_1 \leq n_1$ and $0 \leq k_2 \leq n_2$,

$$\mathbb{P}_{(1,0)}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2)\,.$$

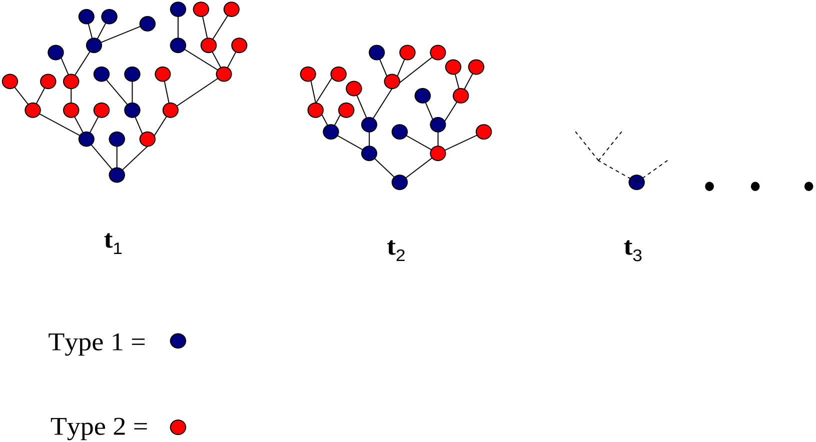
Encoding 2-type forests

Define a 2-type forest,

$$\mathcal{F} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots\},$$

as an infinite sequence of independent 2-type rooted trees, with progeny law (μ_1, μ_2).

- Each vertex $u \in \mathbf{t}_i$ is either of type 1 or type 2.
- The root of each tree is of type 1.
- Vertices of $\mathcal{F}$ are ranked in the breadth first search order.



$\mathbf{t}_1$
$\mathbf{t}_2$
$\mathbf{t}_3$
Type 1 =
Type 2 =

Encoding 2-type forests

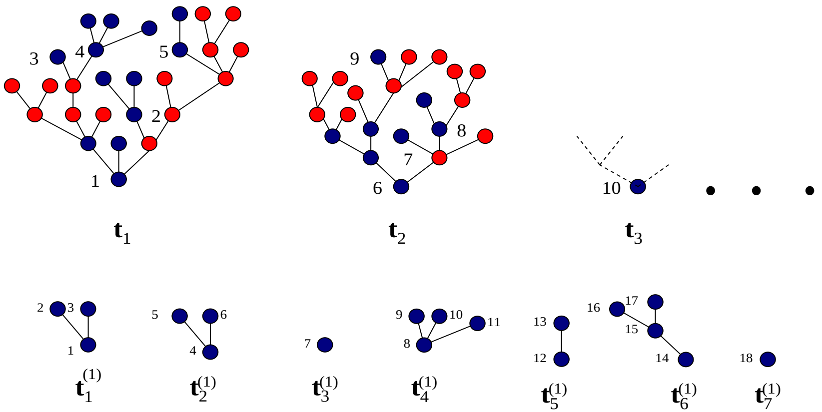
Ordering vertices of type 1 :

- Subtrees of type 1 are ranked according to the breadth first search order of their roots in the forest :

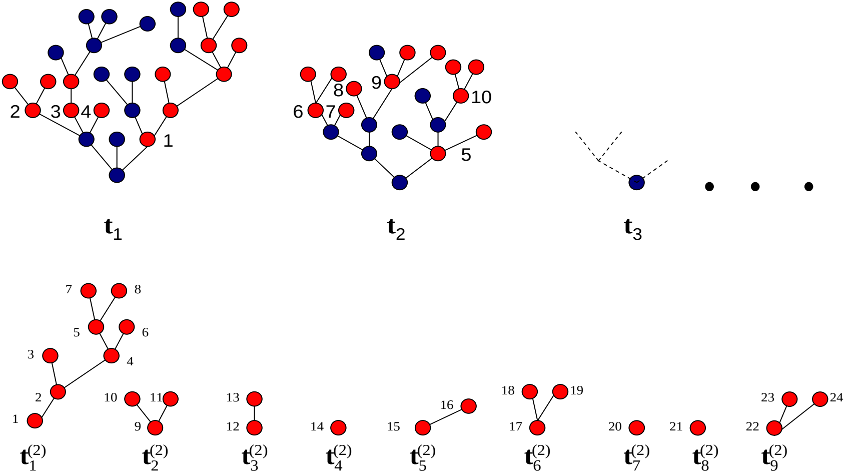
$$\mathbf{t}_1^{(1)}, \mathbf{t}_2^{(1)}, \dots, \mathbf{t}_n^{(1)}, \dots$$

- Then vertices $u_i^{(1)}, \dots, u_j^{(1)}$ of $\mathbf{t}_n^{(1)}$ are ranked according to the 'local' breadth first search order of $\mathbf{t}_n^{(1)}$:

$$\underbrace{u_0^{(1)}, \dots, u_{i_1-1}^{(1)}}_{\mathbf{t}_1^{(1)}}, \underbrace{u_{i_1}^{(1)}, \dots, u_{i_1+i_2-1}^{(1)}}_{\mathbf{t}_2^{(1)}}, \dots\dots$$



1
2
3
4
5
$\mathbf{t}_1$
6
7
8
9
$\mathbf{t}_2$
10
$\mathbf{t}_3$
1
2
3
$\mathbf{t}_1^{(1)}$
4
5
6
$\mathbf{t}_2^{(1)}$
7
$\mathbf{t}_3^{(1)}$
8
9
10
11
$\mathbf{t}_4^{(1)}$
12
13
$\mathbf{t}_5^{(1)}$
14
15
16
17
$\mathbf{t}_6^{(1)}$
18
$\mathbf{t}_7^{(1)}$



1
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$\mathbf{t}_1$
$\mathbf{t}_2$
$\mathbf{t}_3$
$\mathbf{t}_1^{(2)}$
$\mathbf{t}_2^{(2)}$
$\mathbf{t}_3^{(2)}$
$\mathbf{t}_4^{(2)}$
$\mathbf{t}_5^{(2)}$
$\mathbf{t}_6^{(2)}$
$\mathbf{t}_7^{(2)}$
$\mathbf{t}_8^{(2)}$
$\mathbf{t}_9^{(2)}$

Encoding 2-type forests

Let $k_i(u)$ be the number of children of type i of the vertex u.

Then define the integer valued chains $X = (X^{(1)}, X^{(2)})$ and $Y = (Y^{(1)}, Y^{(2)})$ by :

$$X^{(1)}_{n+1} - X^{(1)}_n = k_1(u^{(1)}_n) - 1 \qquad Y^{(1)}_{n+1} - Y^{(1)}_n = k_1(u^{(2)}_n)$$

$$X^{(2)}_{n+1} - X^{(2)}_n = k_2(u^{(1)}_n) \qquad Y^{(2)}_{n+1} - Y^{(2)}_n = k_2(u^{(2)}_n) - 1 \,.$$

Proposition

The chains X and Y are independent random walks in $\mathbb{Z} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \mathbb{Z}$, respectively, with step distributions :

$$P(X_1 = (i,j)) = \mu_1(i+1, j)\,, \quad P(Y_1 = (i,j)) = \mu_2(i, j+1)\,.$$

Encoding 2-type forests

Define

$$T_k = \inf\{n : X_n^{(1)} = -k\} \qquad S_k = \inf\{n : Y_n^{(2)} = -k\}\,.$$

Then,

- $X^{(2)}(T_k)$ is the number of subtrees of type 2 encountered when k subtrees of type 1 have been visited,
- $Y^{(1)}(S_k)$ is the number of subtrees of type 1 encountered when k subtrees of type 2 have been visited.

Therefore, if k_i, $i = 1, 2$ is the total number of subtrees of type i in the first tree $\mathfrak{t}_1$ of the 2-type forest $\mathcal{F}$, then

$$\begin{cases} k_2 = X^{(2)}(T_{k_1}) \\ k_1 = 1 + Y^{(1)}(S_{k_2})\,. \end{cases}$$

Encoding 2-type forests

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Encoding 2-type forests

Let (k_1, k_2) be the smallest solution of

$$(S) \quad \left\{ \begin{array}{l} k_2 = X^{(2)}(T_{k_1}) \\ k_1 = 1 + Y^{(1)}(S_{k_2})\,. \end{array} \right.$$

Proposition

- *k_i, $i = 1, 2$ is the total number of subtrees of type i in $\mathbf{t}_1$.*
- *T_{k_1} is the total number of individuals of type 1 in $\mathbf{t}_1$.*
- *S_{k_2} is the total number of individuals of type 2 in $\mathbf{t}_1$.*
- *$\mathbf{t}_1$ is encoded by the two 2-dimensional chains :*

$$[(X_n^{(1)}, X_n^{(2)}),\, 0 \le n \le T_{k_1}]$$
$$[(Y_n^{(1)}, Y_n^{(2)}),\, 0 \le n \le S_{k_2}]\,.$$

The progeny law

Recall that :

- O_1 : total number of individuals of type 1.
- O_2 : total number of individuals of type 2.
- N_1 : total number of individuals of type 1 whose parent is of type 2.
- N_2 : total number of individuals of type 2 whose parent is of type 1.

$$(S) \quad \begin{cases} k_2 = X^{(2)}(T_{k_1}) \\ k_1 = 1 + Y^{(1)}(S_{k_2}) \, . \end{cases}$$

Then,

$$\mathbb{P}_{(1,0)}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) =$$
$$P(T_{k_1} = n_1, S_{k_2} = n_2 \text{ and } (k_1, k_2) \text{ is the smallest solution of } (S).)$$

The progeny law

Let $(U_k\,, 0 \le k \le k_1)$ and $(V_k\,, 0 \le k \le k_2)$ be independent, integer valued, nondecreasing, with $U_0 = V_0 = 0$ and with cyclically exchangeable increments.

$$(S_{U,V}) \quad \begin{cases} k_1 = r_1 + V_{k_2} \\ k_2 = r_2 + U_{k_1} \end{cases}$$

Theorem (Bivariate ballot Theorem)

Assume that $(S_{U,V})$ admits a solution a.s., then

$$P(\ (k_1, k_2) \text{ is the smallest solution of } (S).) = \frac{k_1 r_2 + k_2 r_1 - r_1 r_2}{k_1 k_2} P(U_{k_1} = k_2 - r_2, V_{k_2} = k_1 - r_1)\,.$$

The progeny law

Apply the biveriate ballot Theorem to $r_1 = 1$, $r_2 = 0$, and to $U_k = X^{(2)}(T_k)$ and $V_k = Y^{(1)}(S_k)$,

$$\mathbb{P}_{(1,0)}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2)$$

$$P(T_{k_1} = n_1, S_{k_2} = n_2 \text{ and } (k_1, k_2) \text{ is the smallest solution of } (S).)$$

$$= \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(S_{k_2}) = k_1 - 1, X^{(2)}(T_{k_1}) = k_2)$$

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$$= \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(S_{k_2}) = k_1 - 1, X^{(2)}(T_{k_1}) = k_2)$$

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$$= \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2)\mu_2^{*n_2}(k_1, n_2 - k_2)\,.$$

The progeny law

More generally, when $\mathbf{Z}_0 = (r_1, r_2)$:

Theorem

Assume that $\mathbf{Z}$ *is irreducible and critical or subcritical and* $\mathbf{Z}_0 = (r_1, r_2)$. *Then for all* $n_1 \geq r_1$ $n_2 \geq r_2$, $r_1 \leq k_1 \leq n_1$ *and* $r_2 \leq k_2 \leq n_2$,

$$\mathbb{P}_{(r_1,r_2)}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - r_1, N_2 = k_2 - r_2) = \frac{r_1 k_2 + r_2 k_1 - r_1 r_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2)\,.$$

The progeny law

Three types :

- A_{ij} = number of individuals of type j whose parent is of type i.

Theorem

Assume that $\mathbf{Z}$ *is irreducible and critical or subcritical and* $\mathbf{Z}_0 = (r_1, r_2, r_3)$. *Then for all* $n_j \geq 1$ *and* $0 \leq k_{ij} \leq n_j$, $j = 1, 2, 3$,

$$\begin{aligned}
&\mathbb{P}(O_1 = n_1, O_2 = n_2, O_3 = n_3, A_{ij} = k_{ij}, i = 1, 2, 3, i \neq j) = \\
&(n_1 n_2 n_3)^{-1}\{r_1[(r_3 + k_{12})k_{23} + (k_{23} + r_2 + k_{12})(r_3 + k_{13})] + \\
&k_{21}[r_3 k_{32} + (k_{23} + r_3 + k_{13})r_2] + k_{31}[r_2 k_{23} + (k_{32} + r_2 + k_{12})r_3]\} \\
&\times \mu_1^{*n_1}(n_1 - k_{21} - k_{31} - r_1, k_{12}, k_{13}) \\
&\times \mu_2^{*n_2}(k_{21}, n_2 - k_{12} - k_{32} - r_2, k_{23}) \\
&\times \mu_3^{*n_3}(k_{31}, k_{32}, n_3 - k_{13} - k_{23} - r_3) \,.
\end{aligned}$$